JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 52, 538–545 (1975)

On the Soliton, Invariant, and Shock Solutions of a Fourth-Order Nonlinear Equation*

K. E. LONNGREN, H. C. S. HSUAN,† AND W. F. AMES‡

College of Engineering, University of Iowa, Iowa City, Iowa 52242

A particular transmission line containing a nonlinear capacitance is shown to satisfy the equation $V_{xxtt} + \omega_0^2 V_{xx} - C_s^{-1}[VC_n(V)]_{tt} = 0$, $\omega_0^2 = (LC_S)^{-1}$. This equation is shown to permit the propagation of solitary waves (solitons). In addition, it admits an invariant (similar) solution under the spiral group. Lastly, we demonstrate two implicit traveling wave solutions that permit the evolution of a discontinuity in the first derivatives (shocks).

1. INTRODUCTION

In recent experiments on the formation and propagation of solitary waves ("solitons") by Kolosick *et al.* [1] and a discontinuity between two states ("shocks") by Lonngren *et al.* [2] on a nonlinear dispersive transmission line, we derived a partial differential equation to characterize the line. Since this equation has not been treated before, we examine three classes of solutions herein and comment on their properties.

In Section 2, we derive our equation from the nonlinear transmission line. Section 3 presents the solitary wave (soliton) solution of the equation. Section 4 examines the possible existence of invariant solutions and demonstrates their existence by employing the spiral group. Section 5 discusses the evolution of discontinuities (shocks) that occur from implicit traveling waves. Section 6 concludes the paper and includes some remarks from the experiments in illustration of the theory.

* Supported in part by the National Science Foundation, Grant No. ENG-7400704.

† Present address: Plasma Physics Laboratory, Princeton University, Princeton, N.J. 08540.

‡ Present address: School of Mathematics, Georgia Institute of Technology, Atlanta, Georgia 30332.

2. Derivation of Equation

Consider an infinitely long transmission line composed of distributed sections; a typical section is shown in Fig. 1. The units of L, $1/C_S$, and C_N are henries per unit length, (1/farads) per unit length and farads per unit length, respectively. The equations that can be written from Fig. 1 are

$$\frac{\partial I}{\partial x} + \frac{\partial[VC_N(V)]}{\partial t} = 0,$$
$$\frac{\partial V}{\partial x} + L\frac{\partial I'}{\partial t} = 0, \qquad \frac{\partial^2 V}{\partial x\,\partial t} + \frac{1}{C_S}(I - I') = 0. \tag{1}$$

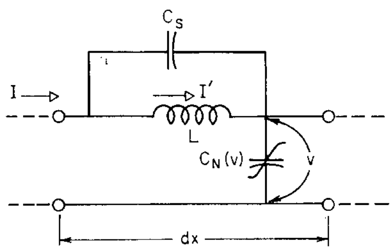

Fig. 1. A typical section of the nonlinear dispersive transmission line.

From set (1), we can derive a "wave" equation for the voltage V:

$$\frac{\partial^4 V}{\partial x^2\,\partial t^2} + \omega_0^2\frac{\partial^2 V}{\partial x^2} - \frac{1}{C_S}\frac{\partial^2[VC_N(V)]}{\partial t^2} = 0, \tag{2}$$

where $\omega_0^2 = 1/LC_S$.

Since $C_N(V)$ will experimentally be a p–n junction diode as discussed in van der Ziel [3], we can assert that $C_N(V) = C_{N0}(V/\bar{V})^{-n}$, where $\bar{V}$ is a normalizing constant and n is some number. We shall show that $0 < n < 1$, which includes many possible experimental values ($n = \frac{1}{3}$ in our experiments).

In dimensionless units, we write Eq. (2) as

$$\frac{\partial^4 \Gamma}{\partial \zeta^2\,\partial \tau^2} + \frac{\partial^2 \Gamma}{\partial \zeta^2} - \frac{\partial^2 \Gamma^{1-n}}{\partial \tau^2} = 0, \tag{3}$$

where

$$\Gamma = V/\bar{V}, \qquad \zeta = (C_{N0}/C_S)^{1/2}\,x, \qquad \text{and} \qquad \tau = \omega_0 t.$$

3. Solitary Wave Solution

A comprehensive review of solitary wave equations and their properties and solutions has been written recently by Scott, Chu, and McLaughlin [4]. In that paper, they listed seven equations that would admit solitary wave solutions. We cite, for example, the Korteweg–deVries, the Sine–Gordon, and the Boussinesq, equations as examples of equations that admit solitary waves.

In order to solve Eq. (3), we first transform to the traveling wave frame $\xi = \zeta \pm M\tau$, where M is the Mach number, $M = u/[\omega_0/(C_{N0}/C_S)^{1/2}]$, and u is the laboratory velocity. Eq. (3) becomes

$$\frac{d^2}{d\xi^2}\left\{M^2\frac{d^2\Gamma}{d\xi^2} + \Gamma - M^2\Gamma^{1-n}\right\} = 0. \tag{4}$$

The integral of the equation within the braces is

$$\frac{M^2}{2}\left(\frac{d\Gamma}{d\xi}\right)^2 + \frac{\Gamma^2}{2} - M^2\frac{\Gamma^{2-n}}{2-n} = 0, \tag{5}$$

where the constant of integration is set equal to zero since a soliton has Γ and $(d\Gamma/d\xi)$ both $\to 0$ as $|\xi| \to \infty$. We now integrate Eq. (5) from $\xi = 0$, where the pulse height is maximum, say Γ_1, to $\xi = \xi$, where $\Gamma = \Gamma$. This integration is facilitated if we let $\Gamma = \gamma^{1/n}$. With this transformation, we write

$$\int_{\gamma_1}^{\gamma} \frac{M(2)^{-1/2}(1/n)\,d\gamma}{[-(\gamma^2/2) + (M^2/(2-n))\,\gamma]^{1/2}} = \int_0^{\xi} d\xi, \tag{6}$$

which can be integrated

$$\begin{aligned} \Gamma = \gamma^{1/n} &= \left\{\frac{M^2}{(2-n)}\left(1 + \cos\frac{n}{M}\xi\right)\right\}^{1/n}, \qquad && \left(\left|\frac{n\xi}{M}\right| \leqslant \pi\right) \\ &= 0, && \left(\left|\frac{n\xi}{M}\right| > \pi\right). \end{aligned} \tag{7}$$

Eq. (7) predicts the shape of the soliton. In addition, we find that the maximum Mach number can be computed in terms of an applied pulse at $\xi = 0$ ($x = 0$ at $t = 0$). In particular $M = (2-n)^{1/2}\,\Gamma_{\text{app}}^{n/2}\,(2)^{1/2}$.

We finally address ourselves to the problem of specifying the limits on n. This can be accomplished by writing Eq. (4) in terms of a potential function $\psi(\Gamma)$:

$$M^2\frac{d^2\Gamma}{d\xi^2} = -\frac{d}{d\Gamma}\left[\frac{\Gamma^2}{2} - M^2\frac{\Gamma^{2-n}}{2-n}\right] = -\frac{d\psi(\Gamma)}{d\Gamma}. \tag{8}$$

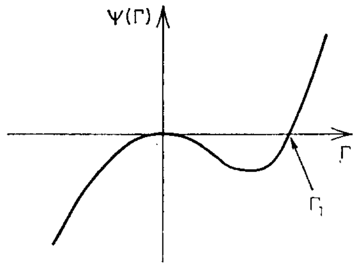

FIG. 2. A sketch of the potential $\psi(\Gamma)$ that allows soliton propagation.

A sketch of $\psi(\Gamma)$ is shown in Fig. 2, which is of the form that allows soliton propagation (i.e., $\psi(\Gamma)\,|_{\Gamma=0} = (d\psi/d\Gamma)\,|_{\Gamma=0} = 0$, and there exists a potential well such that $\psi(\Gamma) < 0$ for $0 < \Gamma < \Gamma_1$, as discussed in Tidman and Krall [5]). Using these conditions, we find $0 < n < 1$.

4. Invariant Solutions

A search for invariant (self-similar) solutions of Eq. (3) can be made by means of one-parameter transformation groups (see Ames [6, 7] for general details). In particular, the nonlinear (spiral) group

$$G: \bar{\tau} = \tau + \ln a, \qquad \bar{\rho} = a^{\alpha}\rho, \qquad \bar{\Gamma} = a^{\beta}\Gamma \tag{9}$$

with parameter a and group invariants

$$\eta = \rho/e^{\alpha\tau}, \qquad f(\eta) = \Gamma/e^{\beta\tau} \tag{10}$$

is applicable. To see this, note that Eq. (3) is constant conformally invariant under G when

$$n\beta = 2\alpha. \tag{11}$$

Thus, with β arbitrary ($\beta \neq 0$), an invariant solution exists where

$$\Gamma(\rho, \tau) = e^{\beta\tau} f(\rho e^{-n\beta\tau/2}). \tag{12}$$

The function f is a solution of the ordinary differential equation

$$\begin{aligned}(n\eta\beta/2)^2 f^{(\mathrm{IV})} &- n\beta^2(1-n)\,\eta f''' + [1 + \beta^2(1-n)^2] f'' \\ &+ \beta^2(1-n)^2 f^{1-n} - n(1-n)\,\beta^2\eta[f^{1-n}]' \\ &+ n^2\beta^2[f^{1-n}]'' = 0,\end{aligned} \tag{13}$$

subject to appropriate boundary conditions. It is by no means certain that (13) has a solution for any prescribed set of boundary conditions.

5. Evolution of Discontinuities (Shocks)

The evolution of discontinuities in solutions of nonlinear hyperbolic equations possessing smooth initial data has been examined by Ludford [8], Lax [9], and Jeffrey [10]. They employ the Riemann invariants and develop comparison theorems that provide upper and lower bounds for the critical time (usually the smallest) of singularity occurrence. Ames [11] found two large classes of second-order equations, at which the previous theories were aimed, that have implicit wave solutions obtainable by differentiation of integrable first-order equations. The advantages of this method, when applicable, are its simplicity and the exactness of the results as opposed to bounds.

To examine the beginning of the evolution of discontinuities in solutions of (2), we first divide by ω_0^2 and assume that $\epsilon = (1/\omega_0^2) \ll 1$. Our calculation is not valid for $\epsilon \sim 1$, which corresponds to the long time shock evolution. Using ϵ as a parameter, the expansion

$$V = V_0 + \epsilon V_1 + \epsilon^2 V_2 + \cdots$$

results in the equation

$$V_{0xx} - L[f(V_0)]_{tt} = 0, \tag{14}$$

where $f(V_0) = V_0 C_N(V_0)$ is written for convenience. For ease in subsequent treatment of the pure initial value problem set

$$\omega = f(V_0), \qquad V_0 = f^{-1}(\omega) = g(\omega), \tag{15}$$

whereupon (14) becomes

$$\omega_{tt} - \frac{\partial}{\partial x}\left[\frac{1}{L} g'(\omega)\, \omega_x\right] = 0, \tag{16}$$

which is one of the general forms treated by Ames [11]. Eq. (16) has non-superposable solutions:

$$\omega = H[x + t(g'(\omega)/L)^{1/2}], \tag{17}$$

$$\omega = G[x - t(g'(\omega)/L)^{1/2}], \tag{18}$$

which are called *implicit traveling waves* (Ames [12]). From (17) or (18), we see that (16), from initial data $H(x)$ or $G(x)$, evolves a waveform whose velocity of motion,

$$[g'(\omega)/L]^{1/2}, \tag{19}$$

depends upon the amplitude, ω, of the wave.

An easy calculation verifies that both ω_x and ω_t may become unbounded as time increases. Thus, from (17)

$$\omega_x = \frac{H'(Lg')^{1/2}}{(Lg')^{1/2} - \frac{1}{2}H'tg''}, \tag{20}$$

where the primes mean the obvious differentiations. Thus, ω_x can be unbounded whenever

$$t = 2\,\frac{(Lg')^{1/2}}{H'g''}, \tag{21}$$

and the critical (minimum) breakdown time is

$$t_{cr} = \min\left[\frac{2(Lg')^{1/2}}{H'g''}\right]. \tag{22}$$

In terms of V_0, the solution (17) becomes

$$V_0 = g\left\{H\left[x + t\left(\frac{dg}{d\omega}\bigg|_{\omega=f(V_0)}\bigg/L\right)^{1/2}\right]\right\}, \tag{23}$$

with a corresponding result for (18). The actual initial condition ($t = 0$) becomes $V_0(x, 0) = g\{H(x)\}$.

For the physical problem of Section 2, where

$$V_0C_N(V_0) = V_0C_{N0}(V_0/\bar{V})^{-n} = AV_0^{1-n} = f(V_0) = \omega, \tag{24}$$

it follows that

$$V_0 = f^{-1}(\omega) = (\omega/A)^{1/(1-n)} = g(\omega). \tag{25}$$

Consequently, (23) becomes

$$f(V_0) = H\left[x + t\left(\frac{1}{L(1-n)\,C_{N0}}\left(\frac{V_0}{\bar{V}}\right)^n\right)^{1/2}\right]. \tag{26}$$

6. Conclusion

In this paper, we have shown that the differential equation that models our transmission line predicts soliton propagation and shock formation. We include two results from experiments by Kolosick *et al.* [1] and Lonngren *et al.* [2] that illustrate both properties.

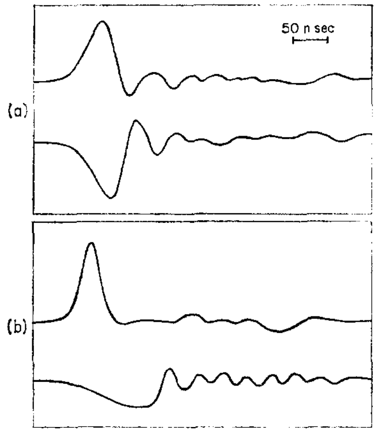

FIG. 3. Response of an experimental transmission line at a fixed distance from the point of excitation by a narrow positive or negative pulse. (a) Linear regime, excitation signal $= \Delta V$; (b) nonlinear regime excitation signal $= 5\Delta V$, where ΔV is in arbitrary units.

In Fig. 3, the property that only a large amplitude compressive pulse will allow soliton formation is illustrated. This is noted by comparing the response to both small and large, compressive and rarefactive excitation pulses. The small amplitude pulse does not produce solitons. Only a large amplitude compressive pulse produces a soliton.

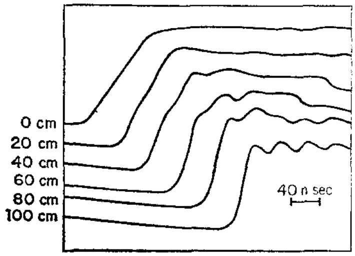

FIG. 4. Experimental observation of the formation of a shock.

In Fig. 4, the formation of a shock is illustrated. We see the characteristic steepening of the wavefront due to the nonlinear elements. Also, the trailing wavetrain of high frequency oscillations ($\sim\omega_0$) is apparent. We cannot rule out the possibility that they may eventually form solitons since no loss mechanism exists in the system.

REFERENCES

1. J. A. KOLOSICK, D. L. LANDT, H. C. S. HSUAN, AND K. E. LONNGREN, Properties of solitary waves as observed on a nonlinear dispersive transmission line, *Proc. IEEE* **62** (1974), 578.
2. K. E. LONNGREN, D. L. LANDT, C. M. BURDE, AND J. A. KOLOSICK, Observation of shocks on a nonlinear dispersive transmission line, *IEEE Trans. Circuits Systems*, **22** (1975), 376.
3. A. VAN DER ZIEL, "Solid State Physical Electronics," Prentice-Hall, Englewood Cliffs, New Jersey, 1968.
4. A. C. SCOTT, F. Y. F. CHU, AND D. W. MCLAUGHLIN, The soliton, a new concept in applied science, *Proc. IEEE* **61** (1973), 1443.
5. D. A. TIDMAN AND N. A. KRALL, "Shock Waves in Collisionless Plasmas," Wiley–Interscience, New York, 1971.
6. W. F. AMES, "Nonlinear Partial Differential Equations in Engineering," Vol. I, Academic Press, New York, 1965.
7. W. F. AMES, "Nonlinear Partial Differential Equations in Engineering," Vol. II, Academic Press, New York, 1972.
8. G. S. S. LUDFORD, On an extension of Riemann's method of integration with applications to one-dimensional gas dynamics, *Proc. Cambridge Philos. Soc.* **48** (1952), 499.
9. P. D. LAX, Development of singularities of solutions of non-linear hyperbolic partial differential equations, *J. Math. Phys.* **5** (1964), 611.
10. A. JEFFREY, The evolution of discontinuities in solutions of homogeneous nonlinear hyperbolic equations having smooth initial data, *J. Math. Mech.* **17** (1967), 331.
11. W. F. AMES, Discontinuity formation in solutions of homogeneous non-linear hyperbolic equations possessing smooth initial data, *Internat. J. Non-linear Mech.* **5** (1970), 605.
12. W. F. AMES, Implicit ad-hoc methods for nonlinear partial differential equations, *J. Math. Anal. Appl.* **42** (1973), 20.